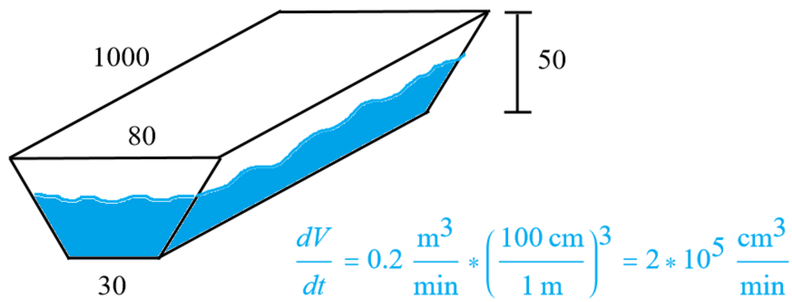
Exercise 27

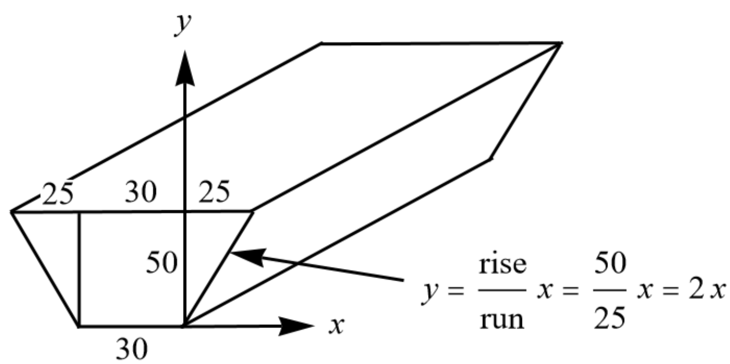
A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

Solution

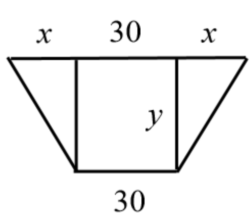
Start by drawing a schematic of the trough at a certain time.



Write an equation for the line representing the trapezoid's edge.



The aim is to find the volume $V(y)$ that water occupies if it's at a height y. In order to do this, find the area of a cross-section and then multiply it by 10 meters (1000 centimeters), the thickness.



This area consists of two triangles and a rectangle.

$$A = \frac{1}{2}xy + 30y + \frac{1}{2}xy$$

$$= xy + 30y$$

Since we want to find dy/dt when $y = 30$, eliminate x in favor of y.

$$A = \left(\frac{y}{2}\right) y + 30y$$

$$= \frac{y^2}{2} + 30y$$

Multiply the area by the thickness, 1000 centimeters, to get the volume.

$$V = 1000A$$

$$= 1000\left(\frac{y^2}{2} + 30y\right)$$

$$= 500y^2 + 30\,000y$$

Take the derivative of both sides with respect to t by using the chain rule.

$$\frac{d}{dt}(V) = \frac{d}{dt}(500y^2 + 30\,000y)$$

$$\frac{dV}{dt} = 1000y \cdot \frac{dy}{dt} + 30\,000 \cdot \frac{dy}{dt}$$

$$200\,000 = 1000(y + 30)\frac{dy}{dt}$$

Solve for dy/dt.

$$\frac{dy}{dt} = \frac{200}{y + 30}$$

Therefore, the rate that the water level is rising when the water is 30 cm deep is

$$\left.\frac{dy}{dt}\right|_{y=30} = \frac{200}{(30) + 30} = \frac{10}{3} \approx 3.333\ \frac{\text{cm}}{\text{min}}.$$